# AMYGDALA $z \mapsto z^2 + c$

A Newsletter of Fractals & M (the Mandelbrot Set)

Issue #21

October 18, 1990

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## IN THIS ISSUE...

In *Polar Coordinates and the Cardioid Body of* $\mathcal{M}$ (page 4), Ralph Walde shows that the "body" of the Mandelbrot set is the classical cardioid figure, and its "head" is a perfect circle.

This seems remarkable, since those features seem at first glance to be subjective, defined only by the beholder's eye — but in fact they are objective, and can be characterized by the limit cycles of their points. Thus: points in the body of $\mathcal{M}$ are those attracted to a single point, and points in the head are those attracted to a two-point cycle.

Stanley S. Miller likes *Steel-Wool Tori* (page 2) — which are not really fractal: they can get pretty kinky, but not *infinitely* kinky. Miller's tori are quite classical, recalling stately 19th century pendula dribbling sand on the floor.

## THE SLIDES (S21)

All four slides accompanying this issue were created by Lee Skinner using FRACTINT (version 13). Three of them have generating functions other than the usual $f_c(z) = z^2 + c$. Later in this issue, Lee shows how to program these functions ( TRIGONOMETRIC IDENTITIES, page 8).

Lee uses an 80386 machine with math chip running at 16 MHz. The images were photographed from the screen of a Super VGA display having 640 x 480 pixels and 256 colors.

All the images have a maximum dwell of 1023.

**4516**: **PLUMAGE.** A dayglow cactus. This image is part of a $\mathcal{J}$-set generated using the exponential function $f_c(z) = \lambda e^z + c$, $\lambda = 0.379 + 0.479i$.
Center = 2.11434 + 0.40892 *i*, magnification = 9.09, aspect ratio = 1.35.

**4517**: **BIGTHREE.** Part of a $\mathcal{J}$-set generated using the function $f_c(z) = \lambda \cosh z + c$, with $\lambda = 1 + .7i$. An unusual feature of this image is the penetration of one family of contours by another.
Center = -1.5708, mag. = 6.94, aspect = 1.35.

**4524**: **CRESCENT.** This is a low-magnification $\mathcal{M}$-set image which has been inverted relative to a circle of radius r = 0.5 with center $p_0 = -0.505$, by transforming each point $p$ of the complex plane to an inverse point $q$. Expressing p in polar coordinates relative to the center $p_0$, we have $p = p_0 + \rho e^{i\theta}$ and $q = p_0 + \frac{1}{\rho} e^{i\theta}$.
Center = -1.5, mag. = 0.67, aspect = 1.33.

### CONTENTS

IN THIS ISSUE... 1
THE SLIDES (S21) 1
SUBMITTING ARTICLES 2
FRACTAL GRAPHICS WORKSHOP 2
STEEL-WOOL TORUS 2
POLAR COORDINATES AND THE CARDIOID BODY OF M 4
AUTHOR 7
TRIVIAL PURSUIT: "AMYGDALA" 7
TRIGONOMETRIC IDENTITIES 8
RECIPE 8
CIRCULATION/RENEWAL 8

**4536**: **STNGRAYP.** This unusual image was generated using Ackermann's generalized exponential function $z^{z^{\cdot^{\cdot^{z}}}}$, where each turn of the iteration "crank" adds one more level of exponentiation. The iteration function is $f_c(z) = c^z$, where we stop iterating when $|z| > 4$.

Center = -1.6826 + -0.0752$i$, mag. = 1.86, aspect = 1.34.

Please note that we're now calling the slide sets "S21", etc. ("Slide set 21") instead of "C20", etc. ("Color slide set 20").

## SUBMITTING ARTICLES

*— Editor*

Here are some guidelines for submitting articles for publication in Amygdala.

1. Type your article as you would like to see it appear in the newsletter. Please do not send handwritten drafts!

In order of preference:

1a. On 3.5" diskette in Macintosh format for FrameMaker, MacWrite, WriteNow, or MS Word; or as a text file. Please also enclose paper copy so I can see your intent.

1b. On 3.5" diskette in IBM format, text file. I have no way to deal with 5.25" diskettes.

1c. Paper copy.

2. Illustrations should be either grayscale (suitable for halftoning) or black/white; not color!

2a. Normally, illustrations will be printed in full column width, so you should make them 3.25" wide, if possible — provided that they're 300 dpi resolution. If they're grainier, make them larger if possible, so that they'll look good when reduced to 300 dpi.

2b. Make sure that you clearly indicate which illustrations go where in the text!

2c. All in all, it's better not to have captions welded into your pictures. Let me put them in ad lib.

2d. I can handle illustrations on diskette in MacPaint, MacDraw, CricketDraw, or Adobe Illustrator formats.

3. Please send along a short biographical note, which I will try to publish in the same issue as your article.

4. Please include your telephone number, in case I have to reach you in a hurry with questions.

## FRACTAL GRAPHICS WORKSHOP

A workshop, *Fractal Graphics as a New Art Form,* was held on August 27-31 at the Glacier Institute in the Taos Ski Valley. Situated in the beautiful Sangre de Cristo mountains of northern New Mexico, the Glacier Institute offers a refreshing and cool setting for summer conferences.

Participants came from seven states around the country. Rollo Silver made a great presentation, with tutorials and lectures which acted as a springboard for cross-fertilization of new ideas and techniques. His Cibachrome images on display were quite inspirational.

Everyone thoroughly enjoyed the interaction and sharing of ideas with people of similar interests yet different experiences. Plans are being made for another fractal graphics workshop next summer during July.

*Dr. Larry Singer,*
*Director, Glacier Institute*

## STEEL-WOOL TORUS

*— Stanley S. Miller*

It's amazing what a computer can do to perk up a harmless branch of 19th century mathematics. Lissajous used a narrow stream of sand pouring from the base of a compound pendulum to produce his curves; I've got an IBM clone with an 8087 co-processor.

Primitive fragments of Lissajous or Bowditch curves were known before the 19th century, of course. That is where they picked up such splendid names as "The Witch of Agnesi" or "The Devil's Curve". What they are, essentially, are parametric equations (XY curves) based on trigonometric functions. If you match $x = \sin t$ against $y = \cos t$ and run $t$ through a cycle from 0.01 to 6.3, you will find yourself looking at a circle. If you then add a second component, for example $x = \sin t - 0.3\cos 3t$ and match it against $y = -0.1\sin 7t$, you will get a more interesting figure, a circle with wiggles.

I came across this world when I purchased a piece of software called CURVES87 for $35 from Bridge Software (POB 118; New Town Branch; Boston, MA 02258). I began experimenting with variations to the parameters and found that with some judicious tinkering I could get whole networks that had never been seen in captivity before. Our mutual friend Dewdney published one of them in the May, 1988 issue of Scientific American, and Science Digest published some more in August 1988.

The secret of this new approach to the old art is to shift the locus of the moving point $t$ off its cycle so that it doesn't get back to the starting point. I do this by setting the first element of the first equation to $\sin 1.01t$ and the first element of the second equation to $\cos 0.99t$. If I then add a second element based on an odd number such as $\sin 3t$ or $\cos 7t$, I can spice up the shape. I can get my wiggles off cycle as well by using $\sin 3.01t$ or $\cos 6.99t$. I apply these second (or third) elements in moderation by slapping a factor such as 0.1 or 0.3 onto them. Once I throw the equation off balance, however, it has to cycle through more rounds, usually 315 cycles ($\pi$ times 100) or 630 cycles ($2\pi$ times 100). Fortunately, the most time these cycles take to compute is a couple of minutes: we are not in the twentieth century world of the Mandelbrot set with its huge drain of computer time, because we are not working pixel by pixel.

### 1. STEEL-WOOL TORUS (Figure 1, next column):

$$x = \sin 0.99t - 0.9\sin 1.01t + 0.6\sin 0.99t - 0.4\sin 2t + 0.1\sin 50t$$

$$y = \cos 0.99t - 0.9\cos 1.01t + 0.6\cos 0.99t - 0.4\cos 2t + 0.1\cos 50t$$

Here is a picture of the steel-wool torus, and two more samples of the Miller Set to try out:

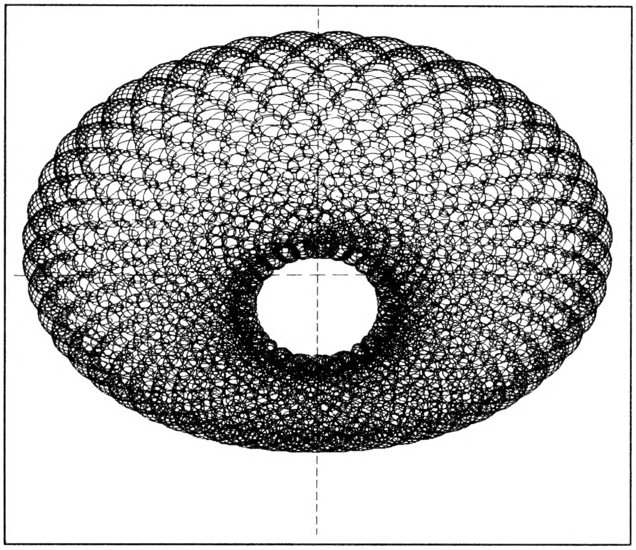


**Figure 1 — STEEL-WOOL TORUS**

**2. MADNESS**, the curve published by Dewdney:

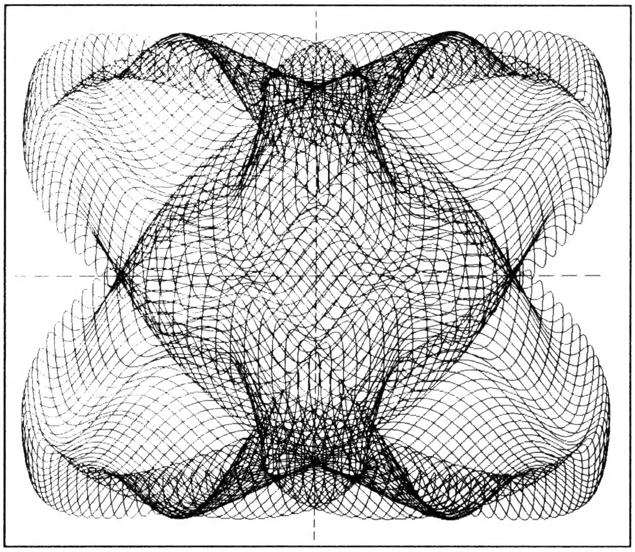


**Figure 2 — MADNESS**

$$x = \sin 0.99t - 0.7\cos 3.01t$$

$$y = \cos 1.01t + 0.1\sin 15.03t$$

Upper left x = -1.7679·5908·98
Upper left y = 1.2072·7858·51
Lower right x = -(Upper left x)
Lower right y = -(Upper left y)
Center: x = y = 0,
H_stretch = 90.2170·1968·32898
V_stretch = 82.4167·6878·19051

Parameter increment size = 0.01

3. **LOLLY'S POLYP**, pictured in Science Digest.

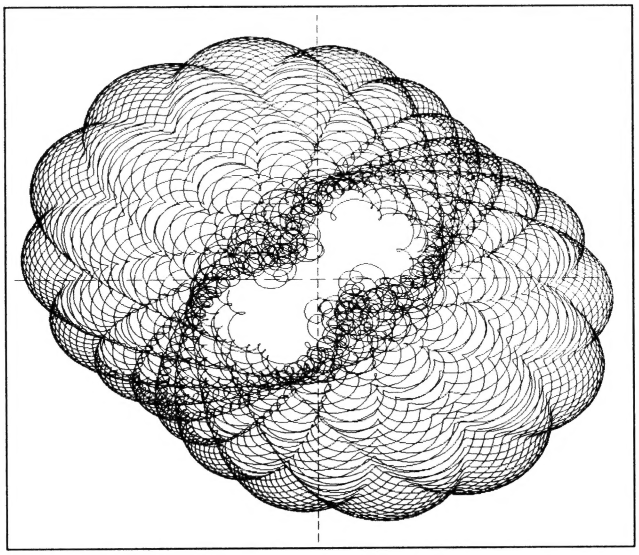


**Figure 3 — LOLLY'S POLYP**

$$x = 1.6\sin 0.99t - 0.9\sin 1.01t$$
$$- 0.4\sin 3t + 0.1\sin 21t$$
$$y = 1.6 \;\cos 0.99t - 0.9\cos 1.01$$
$$- 0.4\cos 3t + 0.1\cos 21t$$

Upper left x = -2.8819·7869·72
Upper left y = 3.0787·2928·39
Lower right x = -(Upper left x)
Lower right y = -(Upper left y)
Center: x = y = 0,
H_stretch = 55.3439·2053·48688
V_stretch = 32.3185·2846·60037
Parameter increment size = 0.015

How many more can you find? I've got closets full of them.

# POLAR COORDINATES AND THE CARDIOID BODY OF $\mathcal{M}$

*Ralph E. Walde*

The article by Frank A. Browne in Amygdala #2 gave a very nice algebraic derivation of equations defining the boundary of the main region of the Mandelbrot set. For readers who are not aware of why this boundary is called a *cardioid*[1], the following tutorial on polar coordinates may be helpful. Using the polar coordinate definition of a cardioid and the relationship of the Mandelbrot set to Julia sets, it will be shown that the boundary of the "body" of the Mandelbrot set is precisely a cardioid and that the boundary of the "head" is precisely a circle. In particular, a trigonometric description of the boundary of the body is the set of complex numbers $\{z_\theta\}$, where the real and imaginary parts of $z_\theta$ are given by:

$$\Re z_\theta = 0.25 + 0.5\cos\theta\,(1 - \cos\theta)$$
$$\Im z_\theta = 0.5\sin\theta\,(1 - \cos\theta)$$

for $0 \le \theta < 2\pi$, and the boundary of the head is given by:

$$\Re z_\theta = -1 + 0.25\cos\theta$$
$$\Im z_\theta = 0.25\sin\theta$$

for $0 \le \theta < 2\pi$.

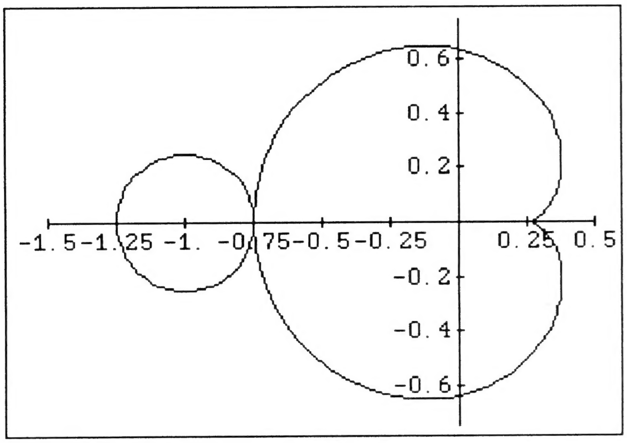


**Figure 1**
**Boundary of the Body and Head of $\mathcal{M}$**

## Polar Coordinates

Cardioids are (or used to be) defined in calculus texts in terms of polar coordinates. Polar coordinates are an alternate way of describing points and curves in the complex plane. Each point in the

---

[1] Pronounced *car' dee oyd*

complex plane is specified by a pair $(R, \theta)$ of real numbers where $R$ is the distance of the point from the origin and $\theta$ is the angle in radians between the positive real axis and the line segment from the point to the origin. $(R, \theta)$ specifies the same point as the complex number $R\cos\theta + iR\sin\theta$.

Thus $(R, \theta) = (0, 0)$ is the origin 0, $(R, \theta) = (1, \pi/2)$ is $i$, $(R, \theta) = (1, \pi)$ is -1, and $(R, \theta) = (\sqrt{2}, \pi/4)$ is $1 + 1i$.

Curves in the complex plane can be described in polar coordinates by specifying a relationship between $R$ and $\theta$. For example, $R = r$ for a positive constant $r$ defines a circle of radius $r$ centered at the origin, consisting of all complex numbers $z_\theta = r\cos\theta + ir\sin\theta$.

The line $ax + by = c$ is described in polar coordinates by $R = \dfrac{c}{a\cos\theta + b\sin\theta}$. To see this, note that since $x = R\cos\theta$ and $y = R\sin\theta$, we have

$$\begin{aligned} c &= ax + by = aR\cos\theta + bR\sin\theta \\ &= R(a\cos\theta + b\sin\theta) \end{aligned}$$

so $\dfrac{c}{a\cos\theta + b\sin\theta} = R$.

Certain curves are far easier to define in polar coordinates than in xy-coordinates. Of greatest interest to us here is the *cardioid*. It is a curve defined in polar coordinates by

$$R = a(1 - \cos\theta)$$

where $a$ is an arbitrary positive constant, which controls the size of the cardioid.

For example, taking $a = 0.5$, the cardioid defined by $R = 0.5(1 - \cos\theta)$ consists of the complex numbers $\{z_\theta\}$, where

$$\Re z_\theta = 0.5\cos\theta(1 - \cos\theta) \qquad [1]$$

$$\Im z_\theta = 0.5\sin\theta(1 - \cos\theta) \qquad [2]$$

for $0 \le \theta < 2\pi$. This particular cardioid is shown in Figure 2.

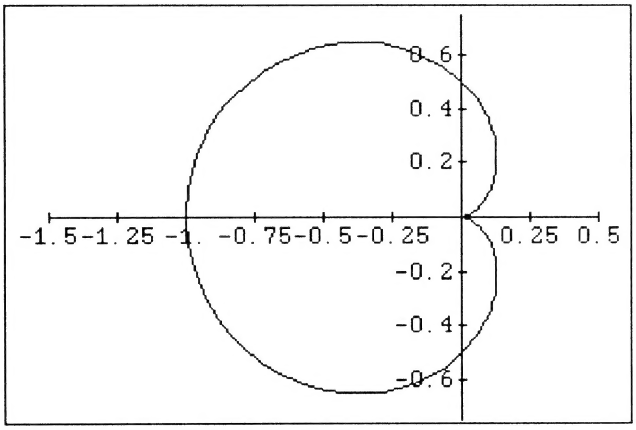


**Figure 2 — Cardioid ($R = 0.5(1 - \cos\theta)$)**

The curve called the *limaçon (lim' uh sun)* is defined by $R = a + b\cos\theta$ for positive $a$ and $b$. It is a generalization of the cardioid, which is a limaçon with $a = b$. Figure 3 shows a limaçon with $a = 0.25$ and $b = 0.75$:

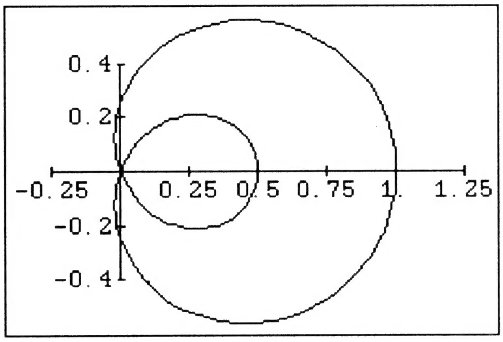


**Figure 3 — Limaçon ($R = 0.25 + 0.75\cos\theta$)**

For any positive constant $a$, $R = a\theta$ defines a spiral, called the *spiral of Archimedes* (Figure 4):

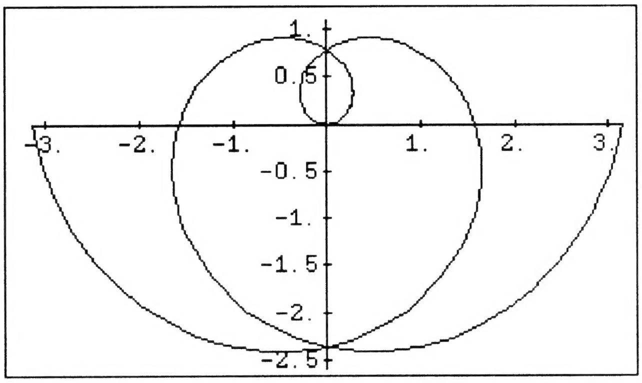


**Figure 4 — Spiral of Archimedes ($R = 0.5\,\theta$)**

and $R = a\cos 2\theta$ defines a *four leaved rose* (Figure 5:

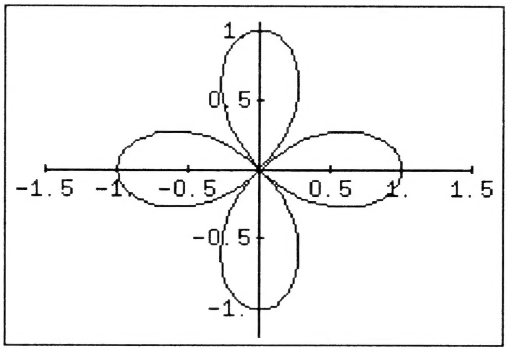


**Figure 5 — Four-leaved Rose ($R = a\cos 2\theta$)**

See older calculus texts for additional examples of curves defined by equations in polar coordinates.

## Regions of the Mandelbrot set

In looking at Julia sets associated with the complex function $f_c(z) = z^2 + c$, it has been found useful to look at regions consisting of those values of the complex number $c$ for which $f_c(z)$ has attractive periodic points of a certain period.

### The Body

Thus, we can define the *body* of the Mandelbrot set to be the region consisting of those complex numbers $c$ for which $f_c(z)$ has an "attractive fixed point", that is, it consists of those $c$ for which there is a complex number $z_1$ with $f_c(z_1) = z_1$ (fixed point) and $|f_c'(z_1)| < 1$ (attractive), where $f_c'(z) = 2z$ is the derivative of $f_c(z)$.

The *boundary* of the body is then just those fixed points for which the limiting condition $|f_c'(z_1)| = 1$ holds, i.e. the set of those complex numbers $c$ for which there is a $z_1$ with $z_1 = z_1^2 + c$ and $1 = f_c'(z_1) = |2z_1|$, i.e. $|z_1| = 1/2$, which means that $z_1$ lies on a circle of radius $1/2$ centered at the origin, so $z_1 = 0.5\cos\theta + 0.5i\sin\theta$, for some real $\theta$ with $0 \le \theta \le 2\pi$.

The corresponding value of $c$ is found using $z_1 = z_1^2 + c$, which gives $c = z_1 - z_1^2$, or, taking real and imaginary parts separately, $\Re c = 0.5\cos\theta - 0.25\cos^2\theta + 0.25\sin^2\theta$, so applying $\sin^2\theta = 1 - \cos^2\theta$, we get:

$$\Re c = 0.25 + 0.5\cos\theta\,(1 - \cos\theta)\ . \qquad [3]$$

The imaginary part is given by:

$$\Im c = 0.5\sin\theta - 0.5\sin\theta\cos\theta, \text{ or:}$$

$$\Im c = 0.5\sin\theta\,(1 - \cos\theta) \qquad [4]$$

We note that these equations [3] and [4] are almost the same as [1] and [2], which define the cardioid with $a = 1/2$ — except that $1/4$ has been added to $\Re c$. Thus we have shown that as $\theta$ varies between 0 and $2\pi$, the fixed point $z_1$ travels around a circle of radius 1/2 centered at the origin, and the complex number $c$ on the boundary of the body of the Mandelbrot set travels around a cardioid with $a = 1/2$ that has been translated a distance $1/4$ to the right on the real axis ( Figure 1, right).

### The Head

In a similar manner we can define the *head* of $\mathcal{M}$ to be the region consisting of those complex numbers $c$ for which $f_c(z)$ has *two* points for its attractive cycle. For a point $c$ on the *boundary* of the head there are two points $z_1$ and $z_2$ with:

$$z_1 = z_2^2 + c \qquad [5]$$

$$z_2 = z_1^2 + c \qquad [6]$$

and $|f_c'(z_1) \cdot f_c'(z_2)| = 1$, that is (since $f_c'(z) = 2z$):

$$|z_1 z_2| = 1/4 \qquad [7]$$

To determine the values of $c$ for which the latter situation holds, subtract [5] from [6] to get

$$z_2 - z_1 = z_1^2 - z_2^2 = (z_1 - z_2)\cdot(z_1 + z_2)$$

or

$$(z_1 - z_2)\cdot(z_1 + z_2 + 1) = 0 \qquad [8]$$

Since $z_1$ and $z_2$ are distinct, $z_1 - z_2$ cannot be zero, so the other factor of [8] must be:

$$z_1 + z_2 + 1 = 0 \qquad [9]$$

or:

$$z_2 = -1 - z_1 \qquad [10]$$

Substituting [10] in [7] we get:

$$1/4 = |z_1 z_2| = |z_1(-1-z_1)| = |-z_1 - z_1^2|$$

Applying [6] and [9]:

$$1/4 = |-z_1 - z_2 + c| = |1+c|.$$

Thus the boundary of the head is the set of $c$ for which $|c-(-1)| = 1/4$, which defines a circle of radius 1/4 centered at $-1$ (Figure 1, left).

## AUTHOR

*Ralph E. Walde*

I am a mathematics professor at Trinity College in Hartford, Connecticut. I am presently teaching a combination of mathematics and computer science courses. I became interested in fractals and chaos a number of years ago and have enjoyed generating a few fractal images on a Macintosh Plus computer with programs written in Pascal. I am currently interested in experimenting with similar programs on the Sun 360 color work stations recently purchased by Trinity.

## TRIVIAL PURSUIT: “AMYGDALA”

I don’t know who sent in the following items, since the signature was cut off the bottom of the copy.

### AMYGDALA...

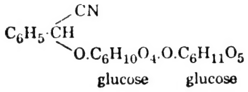
...is a medical term describing a part of the human brain. The following is quoted from Dorland’s Illustrated Medical Dictionary (26th edition, Saunders, Philadelphia, 1985):

*Corpus amygdaloideum* — the amygdaloid body: a small, ovoid complex of nuclei partly covered by the pyriform cortex, within the tip of the temporal lobe, anterior to the inferior horn of the lateral ventricle of the brain; it is part of the limbic system, and is sometimes classified as one of the basal ganglia. The amygdala has olfactory connections, is reciprocally connected to the limbic cortex, and projects fibers to the hippocampus, the septum, the thalamus, and especially to the hypothalamus. Called also *amygdala* and *nucleus amygdalae*.

### AMYGDALIN...

... is a biochemical substance. Its formula is $C_6H_5CH(CN)\text{-}O\text{-}C_{12}H_{21}O_{10}$. It is a cyanogenic glycoside characteristically found in seeds and other plant parts of members of the Rosaceae family, e.g. almonds. The following excerpt is taken from a German catalog of chemical substances: [1]

**Amygdalin.** Amygdaloside; mandelonitrile-$\beta$-gentiobioside; D-mandelonitrile-$\beta$-D-glucosido-6-$\beta$-D-glucoside. $C_{20}H_{27}NO_{11}$; mol. wt. 457.42. C 52.51%, H 5.95%, N 3.06%, O 38.47%. Occurs in seeds of *Rosaceae;* principally in almonds. Obtained by extracting defatted almond meal repeatedly with boiling 95% alcohol. Structure and synthesis: Haworth, Wylam, *J. Chem. Soc.* **123,** 3120 (1923); Kuhn, *Ber.* **56,** 857 (1923); Hudson, *J. Am. Chem. Soc.* **46,** 483 (1924); Zemplén, Kunz, *Ber.* **57,** 1357 (1924). Is split by emulsin: Amygdalase in emulsin produces glucose and mandelonitrile glucoside which are converted by prunase in emulsin to form benzaldehyde, glucose (2 mols.), and HCN. Hydrolysis with hot dilute mineral acids gives same end products. Cf. Viehoever, Mack “Biochemistry of Amygdalin,” *Am. J. Pharm.* **107,** 397–450 (1935).



$$C_6H_5{\cdot}CH(CN)\,O.\underbrace{C_6H_{10}O_4}_{\text{glucose}}.O.\underbrace{C_6H_{11}O_5}_{\text{glucose}}$$

Trihydrate, $C_{20}H_{27}NO_{11}.3H_2O$, orthorhombic columns from water, m. 200°. m. about 220° when anhydrous. The once melted and solidified substance melts again at 125–130°. $[\alpha]_D^{20}$ −42° (c = 1, anhydrous basis). One gram of the trihydrate dissolves in 12 ml. water, in 900 ml. alcohol, in 11 ml. boiling alcohol. Very soluble in boiling water, insoluble in ether. pH of satd. water soln. about 7.

### *AMYGDALINE* and *AMYGDALOID...*

... both mean “resembling an almond”.

### AMYGDALOIDS...

... are almond-shaped stones, used for seals with deeply engraved icons, in the middle and late Minoic times.

Perhaps this little “amygdaclopedia” can be extended.

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[1] Here, I’ve substituted the entry from The Merck Index of Chemicals and Drugs (7th edition (Merck & Co., Inc., Rahway, N.J. U.S.A. 1960). — *Ed.*

## TRIGONOMETRIC IDENTITIES

— *Lee Skinner*

The following trig identities are invaluable for coding fractals that use complex-valued transcendental functions.

$$e^{x+iy} = e^x \cdot \cos y + i \cdot e^x \cdot \sin y$$
$$= (\cosh x + \sinh x) \cdot (\cos y + i \cdot \sin y)$$

$$\sin(x+iy) = \sin x \cdot \cosh y + i \cdot \cos x \cdot \sinh y$$

$$\cos(x+iy) = \cos x \cdot \cosh y - i \cdot \sin x \cdot \sinh y$$

$$\sinh(x+iy) = \sinh x \cdot \cos y + i \cdot \cosh x \cdot \sin y$$

$$\cosh(x+iy) = \cosh x \cdot \cos y + i \cdot \sinh x \cdot \sin y$$

$$\log(x+iy) =$$
$$= \frac{1}{2} \cdot \log(x^2+y^2) + i \cdot \left(\operatorname{atan}\frac{y}{x} + 2k\pi\right),$$
$$k = 0, \pm 1, \pm 2, \ldots$$

$$\tan(x+iy) =$$
$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \cdot \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\tanh(x+iy) =$$
$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i \cdot \frac{\sin 2y}{\cosh 2x + \cos 2y}$$

$$z^z = e^{z \log z}$$

Comment by R.S.:

Lee used these identities in programming the images shown in the slides (S21).

Watch out for multivaluedness when iterating functions involving $\log z$, such as

$$f(z) = z^{2.5} + c = e^{2.5 \cdot \log z} + c$$

The "fracture lines" or "slip planes" apparent in images based on multivalued functions are artifacts of the inconsistent choice of values of the function on the two sides of the discontinuity.

I don't have a solution for this problem. Georg Friedrich Bernhard Riemann might, but he dead. Anybody have an idea of how to render Riemann surfaces graphically nicely?

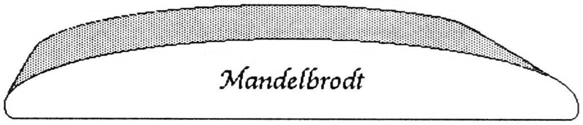
## RECIPE

— *Leonard Herzmark*

This is from my wife's grandmother: Mandelbrodt!!

3 eggs
1 cup sugar
1 cup Mazola or
peanut oil
3.5 cups flour
1 tsp. vanilla
1 lb. walnuts or pecans! [1]
2 tsp. baking powder
1 - 2 cups white raisins

Beat eggs & sugar. Continue beating while adding oil, vanilla, nuts, raisins, flour, & baking powder. Oil baking sheet — flour hands well, roll loaves as cylinders 2" in diameter, then press into elliptical cylinders with flat bottom, minor axis about 1". Bake loaves 0.75 hr at 350 ° F. Slice into cakes about 0.75" thick, separate, bake 15 min longer. Ess, ess, mein kind. Enjoy!



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[1] I can't see how it can be *Mandelbrodt* without *almonds* — but who am I to argue with somebody's grandmother? — *Ed.*

## CIRCULATION

As of October 15, 1990 Amygdala has 735 subscribers, 284 of whom have the supplemental color slide subscription.

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